MATH 5061 Lecture 1 (Jan 13)

Assessment: Biweekly HW 50% ; Take-home Final 50%

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Please refer to course webpage for more details / announcements.

Q: What is "Riemannian Geometry"?

$\boxed{\text{Undergrad. Diff. Geom. } (\sim \text{MATH } 4030)}$ $\xrightarrow{\text{continuation}}$ $\boxed{\text{Riemannian Geometry (MATH 5061)}}$

Curves / surfaces in $\mathbb{R}^2$ or $\mathbb{R}^3$ (curvatures)

"higher dim."

"intrinsic geometry"

geometry / topology / calculus / analysis, PDEs

(M^n, g)

M^n: abstract (smooth) manifold

g: Riemannian metric

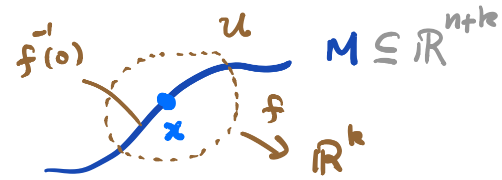
We begin with the theory of (smooth C^∞) manifolds.

(Ref: F. Warner "Foundations of Differential Manifolds and Lie Groups"

J. Lee "Introduction to Smooth Manifolds")

§ Submanifolds of $\mathbb{R}^N$

Def$^{\underline{n}}$: $M \subseteq \mathbb{R}^{n+k}$ is an n-dim'l submanifold of $\mathbb{R}^{n+k}$ (of class C^p) if $\forall\, x \in M$, $\exists$ nbd. $x \in \mathcal{U} \subseteq \mathbb{R}^{n+k}$ and a C^p-map $f : \mathcal{U} \to \mathbb{R}^k$ s.t.



(i) $df_q : \mathbb{R}^{n+k} \to \mathbb{R}^k$ is onto $\forall q \in \mathcal{U}$ (i.e. f is a submersion)

(ii) $\mathcal{U} \cap M = f^{-1}(0)$

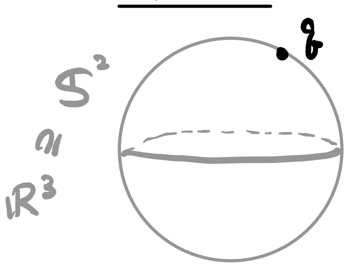
<u>Idea:</u> $M \overset{\text{locally}}{=\!=}$ "regular" zero set of a C^p (vector-valued) function

Here, $k = \operatorname{codim} M$, $n = \dim M$.

When $k = 1$, we say M is a hypersurface.

<u>Examples of submfd in $\mathbb{R}^N$</u>

(a) <u>Sphere</u> $\mathbb{S}^n := \{ (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_0^2 + \dots + x_n^2 = 1 \}$



$f(x,y,z) = x^2 + y^2 + z^2 - 1$

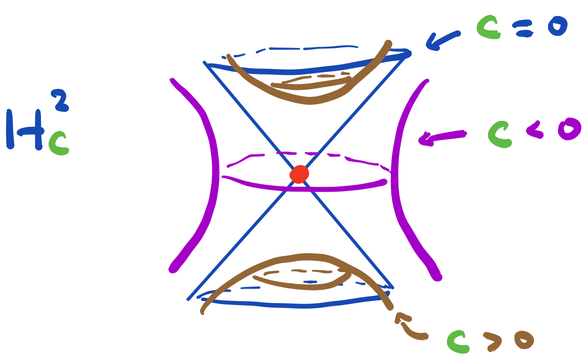
$\Rightarrow f : \mathbb{R}^3 \to \mathbb{R}$ C^∞ and $f^{-1}(0) = \mathbb{S}^n$

$df_q = (2x, 2y, 2z) \neq 0 \quad \forall q \in \mathbb{S}^n$

(b) <u>Hyperboloid</u> $H^n_c := \{ (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_0^2 - x_1^2 - \dots - x_n^2 = c \}$

is a C^∞-submfd in $\mathbb{R}^{n+1}$ when $0 \neq c \in \mathbb{R}$.

<u>$n = 2$</u> :



(c) <u>Torus</u> $T^n := \{ (z_1, \dots, z_n) \in \mathbb{C}^n \mid |z_1|^2 + \dots + |z_n|^2 = 1 \}$

is a C^∞-submanifold of $\mathbb{C}^n \cong \mathbb{R}^{2n}$

($\dim = n$, $\operatorname{codim} = n$)

(d) $SO(n) := \{ A \in M_n(\mathbb{R}) \mid A^t A = I,\ \det A = 1 \}$

is a C^∞-submfd of $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ $\left(\dim SO(n) = \frac{n(n-1)}{2} \right)$

Why? $GL_n^+(\mathbb{R}) := \{ A \in M_n(\mathbb{R}) \mid \det A > 0 \} \underset{\text{open}}{\subseteq} M_n(\mathbb{R})$

Define: $f : GL_n^+(\mathbb{R}) \longrightarrow \mathrm{Sym}(n) := \{ A^t = A \} \underset{\text{linear subsp.}}{\subseteq} M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$

$$f(A) := A^t A - I$$

Note: • $f^{-1}(0) = SO(n)$

• At any $A \in SO(n)$, $df_A(B) = A^t B + B^t A$

is an onto map to $\mathrm{Sym}(n)$

$\left[\text{Given } S \in \mathrm{Sym}(n),\ df_A\left(\frac{AS}{2}\right) = A^t\left(\frac{AS}{2}\right) + \left(\frac{SA^t}{2}\right) A = S. \right]$

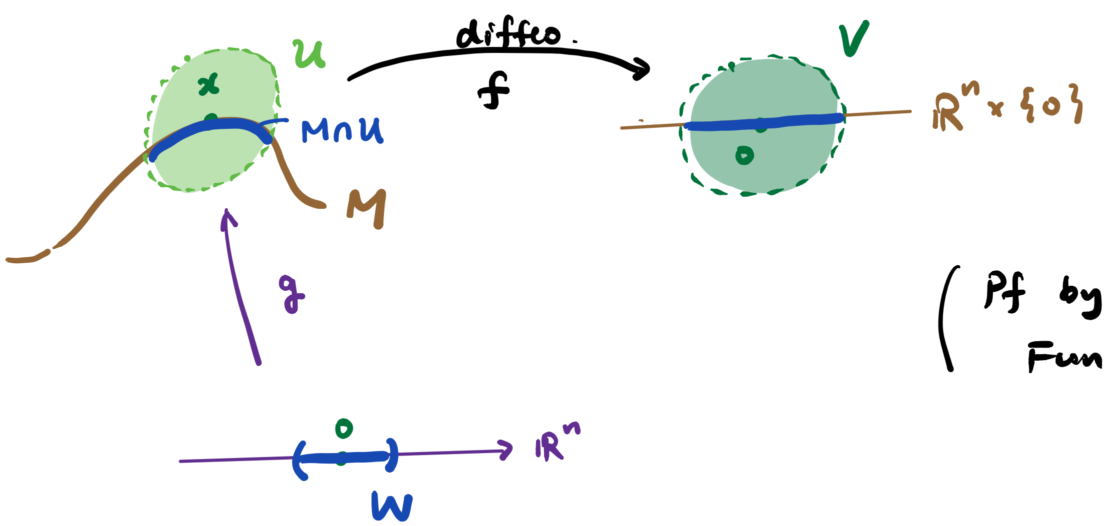
Prop: TFAE:

(i) $M^n \subseteq \mathbb{R}^{n+k}$ is a C^p-submanifold

(ii) $\forall x \in M$, $\exists$ nbd $x \in \mathcal{U} \subseteq \mathbb{R}^{n+k}$, $0 \in V \subseteq \mathbb{R}^{n+k}$

and a C^p-diffeomorphism $f : \mathcal{U} \to V$

s.t. $f(\mathcal{U} \cap M) = V \cap (\mathbb{R}^n \times \{0\})$



(Pf by Implicit Function Theorem)

(iii) $\forall x \in M$, $\exists$ nbd $x \in \mathcal{U} \subseteq \mathbb{R}^{n+k}$ and $0 \in W \subseteq \mathbb{R}^n$

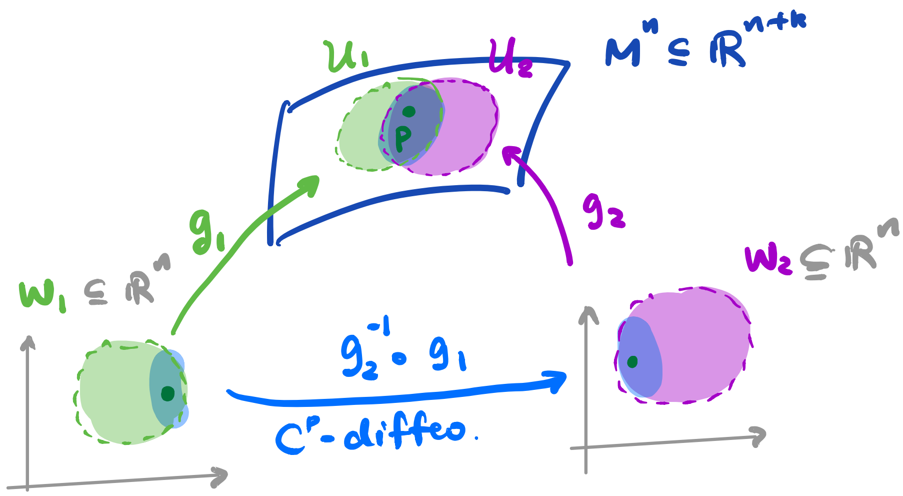
and a C^p-map $g : W \to \mathbb{R}^{n+k}$

s.t. g is a homeomorphism onto its image $g(W) = M \cap \mathcal{U}$

with dg_0 is 1-1. ("local parametrization / chart")

Remark: For any two such charts $g_i : W_i \to \mathbb{R}^{n+k}$, $i=1,2$, as in (iii)

then the transition maps $g_2^{-1} \circ g_1$ is a C^p-diffeomorphism



§ Abstract Manifolds

Idea: "n-manifolds" $\overset{\text{"locally"}}{\approx}$ open subsets of $\mathbb{R}^n$

described by "compatible" charts into $\mathbb{R}^n$

ASSUME: M Hausdorff, "paracompact" topological space

[$\Rightarrow$ $\exists$ "partition of unity"]

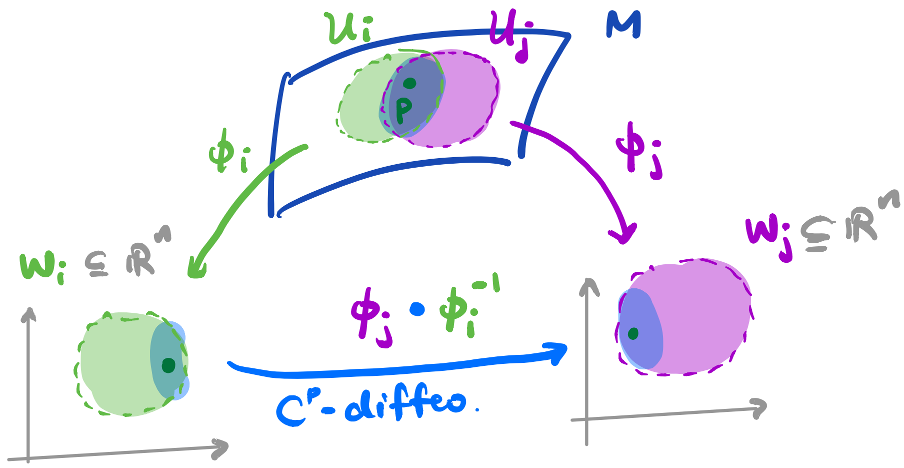
Def$^{\underline{n}}$: A C^p-atlas on M is a collection of charts $\{(U_i, \phi_i)\}_{i \in I}$

s.t. (i) $\{U_i\}_{i \in I}$ forms an open cover of M

(ii) $\phi_i : U_i \to W_i \overset{\text{open}}{\subseteq} \mathbb{R}^n$ are homeomorphisms $\forall i \in I$

and the transition maps

$$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j) \text{ are } C^p\text{-diffeo.}$$



Def$^{\underline{n}}$: $\{(U_i, \phi_i)\}_{i \in I} \sim \{(V_j, \psi_j)\}_{j \in J}$ if their union is an atlas

Def$^{\underline{n}}$: An equivalence class of C^p-atlas on M is called a differentiable structure (of class C^p) on M

A differential manifold consists of a Hausdorff, paracompact topological space M together with an atlas $\{(U_i, \phi_i)\}_{i \in I}$.

Remark: M connected $\Rightarrow$ $\dim M = n$ well-defined

(by "invariance of domain")

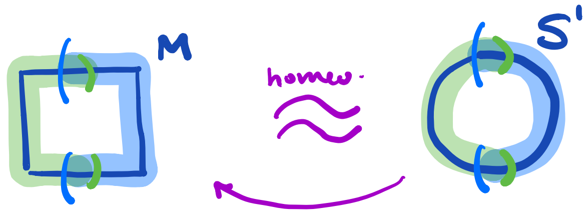
ASSUME: M^n connected, smooth (i.e. C^∞) manifold

Def$^{\underline{n}}$: $N \subseteq M^n$ is a submanifold if $\forall p \in N$, $\exists$ chart (U, ϕ) of M s.t. $p \in U$ and $\phi(N \cap U) \subseteq \mathbb{R}^n$ is a submfd.

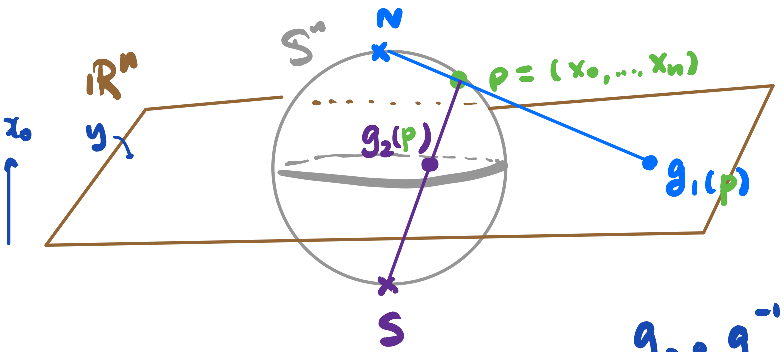
Examples of abstract manifolds

(a) submfds of $\mathbb{R}^{n+k}$

(b) $\exists$ C^∞- structure on a square (not inherited from $\mathbb{R}^2$)



(c) $S^n \subseteq \mathbb{R}^{n+1}$ can be covered by only two charts:

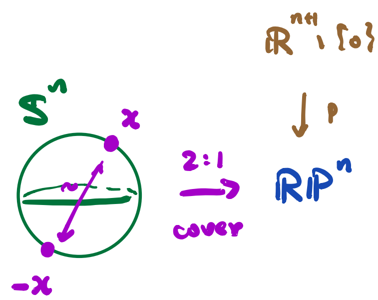


$$g_1(x_0, \dots, x_n) = \frac{(x_1, \dots, x_n)}{1 - x_0} \qquad p \neq N$$

$$g_2(x_0, \dots, x_n) = \frac{(x_1, \dots, x_n)}{1 + x_0} \qquad p \neq S$$

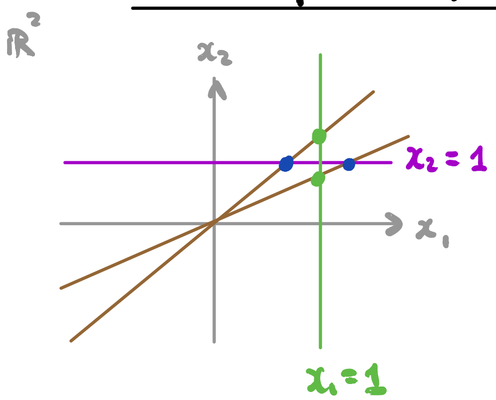
$$g_2 \circ g_1^{-1}(y) = \frac{y}{\|y\|^2}$$ diffeo. on $\mathbb{R}^n \setminus \{0\}$.

(d) Real Projective Space $\mathbb{RP}^n$.



$$\mathbb{RP}^n := \mathbb{R}^{n+1} \setminus \{0\} \Big/ \underset{(\lambda \neq 0)}{x \sim \lambda x} = S^n / \{\pm 1\}$$

A description of C^∞- structure on $\mathbb{RP}^1$:



$$\Phi_1 : \{x_1 \neq 0\} \to \mathbb{R} \qquad \Phi_1(x_1, x_2) = \frac{x_2}{x_1}$$

$$\Phi_2 : \{x_2 \neq 0\} \to \mathbb{R} \qquad \Phi_2(x_1, x_2) = \frac{x_1}{x_2}$$

Note: $\Phi_i(\lambda x_1, \lambda x_2) = \Phi_i(x_1, x_2) \quad (\lambda \neq 0)$

$\rightsquigarrow$ Φ_i are well-defined on $\mathbb{RP}^1$.

and they form a chart on $\mathbb{RP}^1$

(Ex: check this!)

(e) Replace $\mathbb{R}$ by $\mathbb{C}$ $\rightsquigarrow$ Complex Projective Space $\mathbb{CP}^n$

(dim $= 2n$)

Def$^{\underline{n}}$: M is orientable if $\exists$ atlas $\{(U_i, \phi_i)\}_{i \in I}$ s.t.

all transition maps are orientation-preserving

[i.e. $\det(d(\phi_j \circ \phi_i^{-1})) > 0$].

Examples: S^n is orientable BUT $\mathbb{RP}^n$ is NOT when n is even.

§ Smooth Maps between manifolds

Let M^m, N^n be smooth manifolds.

Def$^{\underline{n}}$: A cts map $f: M \to N$ is smooth

if $\forall x \in M$, $\exists$ charts (U, ϕ) for $x \in M$

and chart (V, ψ) for $f(x) \in N$

s.t. $\psi \circ f \circ \phi^{-1} : \phi(U) \to \psi(V)$ is smooth.

i.e.

$$\begin{tikzcd} M \supseteq \overset{x}{U} \arrow[r, "f"] \arrow[d, "\phi"'] & \overset{f(x)}{V} \subseteq N \arrow[d, "\psi"] \\ \mathbb{R}^m \supseteq \phi(U) \arrow[r, "\psi \circ f \circ \phi^{-1}"'] & \psi(V) \subseteq \mathbb{R}^n \end{tikzcd}$$

Example: $M^n \subseteq \mathbb{R}^{n+k}$ submfd and $F: \mathbb{R}^{n+k} \to \mathbb{R}$ smooth

$\Rightarrow$ $F|_M : M \to \mathbb{R}$ is a smooth map between manifolds

Def$^{\underline{n}}$: A smooth map $f: M \to N$ is called an immersion at $p \in M$

if $\exists$ charts $\overset{p}{(U, \phi)}$ for M, $\overset{f(p)}{(V, \psi)}$ for N s.t.

$d(\psi \circ f \circ \phi^{-1})$ is 1-1 at $\phi(x)$.

Remark: submersion / local diffeo. if it is onto / bijective

Def^{n}: $f: M \to N$ diffeomorphism if f is bijective and both f, f^{-1} are smooth.

Exercise: $\mathbb{CP}^1 \overset{\text{diffeo.}}{\cong} S^2$.